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Inequalities for Modified Bessel Functions

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Abstract. A sequence of sharp versions of the inequality $I_{\nu+1}(x) < I_\nu(x)$, $\nu > -\frac{1}{2}$, $x > 0$, is established.

The modified Bessel function of the first kind I_ν is real-valued for ν real on the domain $x > 0$, and it is positive for $\nu \geqq -1$ on the same domain (see Abramowitz-Stegun [1] for standard properties of special functions). The inequality

$$(1) \qquad I_{\nu+1}(x) < I_\nu(x),$$

where $\nu > -\frac{1}{2}$ and $x > 0$, was established by Soni [8] in 1965. Results that are stronger than (1) for $\nu \geqq 0$ have been given by Jones [3], Cochran [2], and Reudink [7]. Thus, Jones proved that

$$(2) \qquad I_\mu(x) < I_\nu(x)$$

for $\mu > \nu \geqq 0$ and $x > 0$, while Cochran established the inequality $\partial I_\nu(x)/\partial\nu < 0$ for $\nu \geqq 0$ and $x > 0$. Reudink, apparently unaware of the work of the previous authors, proved in a different way that $\partial I_\nu(x)/\partial\nu < 0$ for $\nu > 0$ and $x > 0$.

We observe first that (1) holds for $\nu \geqq -\frac{1}{2}$. Indeed, with $x > 0$, we have

$$I_{-1/2}(x) - I_{1/2}(x) = (2/(\pi x))^{1/2} e^{-x} > 0.$$

In the present note, we prove two propositions. The first one contains a rather modest but easily proved result that strengthens (1) for $\nu > 0$. The second proposition gives a sequence of progressively sharper lower bounds of $I_\nu(x)$ that converge monotonically to $I_\nu(x)$.

PROPOSITION 1. *Let $\nu \geqq -1$ and $x > 0$. Then*

$$(3) \qquad (1 + \nu/x) I_{\nu+1}(x) < I_\nu(x).$$

Proof. The series representation for I_ν is

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\,\Gamma(k + \nu + 1)}.$$

Setting $k + 1 = j$ gives

$$(4) \qquad I_\nu(x) = \left(\frac{x}{2}\right)^{\nu-2} \sum_{j=1}^{\infty} \frac{(x/2)^{2j}}{j!\,\Gamma(j + \nu + 1)} j(j + \nu).$$

The average of these two expressions is

$$(5) \qquad I_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^{\nu-2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\,\Gamma(k + \nu + 1)} \left[\left(\frac{x}{2}\right)^2 + k(k + \nu)\right].$$

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Replacing ν by $\nu + 1$ in (4), multiplying by $(1 + \nu/x)$, and subtracting the resulting expression from (5) gives

$$I_\nu(x) - \left(1 + \frac{\nu}{x}\right) I_{\nu+1}(x) = \frac{1}{2}\left(\frac{x}{2}\right)^{\nu-2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\, \Gamma(k+\nu+1)} \left(\frac{x}{2} - k\right)^2 > 0,$$

which proves (3).

This result was established by discarding an infinite series of nonnegative terms. A sharp version of (3) results from retaining a finite number of these terms.

As a preparation for Proposition 2, we define two sequences of functions $\{G_{\nu,k}\}$ and $\{H_{\nu,k}\}$, $k = 0, 1, 2, \cdots$, by*

$$G_{\nu,k}(x) = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{2(2\nu+1)_{j-1}(\nu+j)}{(2\nu+k+1)_j} I_{\nu+j}(x) \tag{6}$$

and

$$H_{\nu,k}(x) = G_{\nu,k}(x) + \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^{\nu} e^{-x}, \tag{7}$$

where $\nu > -\frac{1}{2}$ and $x > 0$.

We note that the inequality $G_{\nu,k}(x) < H_{\nu,k}(x)$ follows from these definitions.

PROPOSITION 2. *Let* $\nu > -\frac{1}{2}$ *and* $x > 0$. *Then*

(i) $$0 < H_{\nu,k}(x) < H_{\nu,k+1}(x) < I_\nu(x), \qquad k \geqq 0,$$

(ii) $$H_{\nu,k}(x) \sim I_\nu(x), \qquad x \to 0, \quad k \geqq 0,$$

(iii) $$I_\nu(x) - H_{\nu,k}(x) \sim \frac{(2\nu+1)_k}{(2x)^k} I_\nu(x), \qquad x \to \infty, \quad k \geqq 0,$$

and

(iv) $$\lim_{k\to\infty} H_{\nu,k}(x) = I_\nu(x).$$

Proof. Our proof is based on an expansion of the confluent hypergeometric function in terms of modified Bessel functions. The following expression follows from Luke [4, p. 48]:

$${}_1F_1(a; c; z) = \Gamma(a + \tfrac{1}{2}) e^{z/2} \left(\frac{4}{z}\right)^{a-1/2} \cdot \left[I_{a-1/2}\left(\frac{z}{2}\right) + \sum_{j=1}^{\infty} \frac{2(j + a - \frac{1}{2})(2a)_{j-1}(2a-c)_j}{j!\, (c)_j} I_{j+a-1/2}\left(\frac{z}{2}\right) \right].$$

By letting $k \geqq 0$ be an integer and putting $a = \nu + \frac{1}{2}$, $c = 2\nu + k + 1$, and $z = 2x$, we find from this expression and (6) that

$$I_\nu(x) - G_{\nu,k}(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^{\nu} e^{-x}\, {}_1F_1(\nu + \tfrac{1}{2}; 2\nu + k + 1; 2x). \tag{8}$$

But ${}_1F_1(a; c; z) > 1$ for $a > 0$, $c > 0$, $z > 0$. We conclude therefore from (8) and (7) that $I_\nu(x) > H_{\nu,k}(x)$.

* We adopt the convention $\sum_{k=m}^{n} \alpha_k = 0$ for $n < m$ and the notation $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha + n - 1)$, $n \geqq 1$, $(\alpha)_0 = 1$.

From the contiguous recurrence relations for the confluent hypergeometric function, we find that

$$ {}_1F_1(a; c; z) - {}_1F_1(a; c + 1; z) = \frac{az}{c(c+1)}\, {}_1F_1(a + 1; c + 2; z). $$

From this recurrence relation and (8), we get

$$ G_{\nu,k+1}(x) - G_{\nu,k}(x) = \frac{2(2\nu + 1)}{\Gamma(\nu + 1)(2\nu + k + 1)_2} \left(\frac{x}{2}\right)^{\nu+1} e^{-x}\, {}_1F_1(\nu + \tfrac{3}{2}; 2\nu + k + 3; 2x). $$

Since the right-hand side of this equality is positive, we conclude that

$$ H_{\nu,k+1}(x) - H_{\nu,k}(x) = G_{\nu,k+1}(x) - G_{\nu,k}(x) > 0. $$

This establishes (i) since $H_{\nu,0}(x) > 0$.

By using the first two terms in the series expansions of $I_\nu(x)$, e^{-x}, and ${}_1F_1(\nu + \frac{1}{2}; 2\nu + k + 1; 2x)$, we find from (8) that the asymptotic behavior of $G_{\nu,k}(x)$ as $x \to 0$ is

$$ G_{\nu,k}(x) \sim \frac{2k}{(2\nu + k + 1)\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^{\nu+1}, \qquad k \geqq 0. $$

From (7), we therefore get

$$ H_{\nu,k}(x) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^{\nu} $$

as $x \to 0$ for $k \geqq 0$. This establishes (ii).

We next apply the asymptotic expansion of ${}_1F_1(\nu + \frac{1}{2}; 2\nu + k + 1; 2x)$ as $x \to \infty$ to (8) and use the duplication formula for the Γ-function to establish the asymptotic relation

$$ I_\nu(x) - G_{\nu,k}(x) \sim \frac{(2\nu + 1)_k}{(2x)^k} \frac{e^x}{(2\pi x)^{1/2}} $$

as $x \to \infty$ and $k \geqq 0$. Statement (iii) now follows from (7) and the asymptotic expansion for $I_\nu(x)$ as $x \to \infty$.

Statement (iv) follows via relations (8) and (7) from the observation that

$$ \lim_{c \to \infty} {}_1F_1(a; c; z) = 1. $$

We proceed to compare the inequality $H_{\nu,k}(x) < I_\nu(x)$ with (1), (2), and (3).

From the definition of $G_{\nu,k}$ in (6), we find $G_{\nu,1}(x) = I_{\nu+1}(x)$. Hence, the inequality $H_{\nu,k}(x) < I_\nu(x)$ is sharper than (1) for all $k \geqq 1$.

Let $\mu > \nu \geqq 0$ be fixed. We find then from (ii) that the inequality $H_{\nu,k}(x) < I_\nu(x)$ is sharper than (2) for all $k \geqq 0$, provided x is sufficiently close to 0. The asymptotic behavior as $x \to \infty$ of $I_\nu(x) - I_\mu(x)$ is

$$ I_\nu(x) - I_\mu(x) \sim \frac{\mu^2 - \nu^2}{2x} I_\nu(x). $$

A comparison with (iii) shows that $H_{\nu,k}(x) < I_\nu(x)$ is a sharper inequality than (2) for $k \geqq 2$ and all sufficiently large values of x.

In order to effect a comparison with (3), we put $k = 2$ in (6) to get

$$G_{\nu,2}(x) = 2\frac{\nu+1}{\nu+\frac{3}{2}} I_{\nu+1}(x) - \frac{\nu+\frac{1}{2}}{\nu+\frac{3}{2}} I_{\nu+2}(x).$$

The recurrence relation

$$I_{\nu+2}(x) = I_\nu(x) - \frac{2(\nu+1)}{x} I_{\nu+1}(x) \tag{9}$$

then gives

$$I_\nu(x) - G_{\nu,2}(x) = \frac{2(\nu+1)}{\nu+\frac{3}{2}}\left[I_\nu(x) - \left(1 + \frac{\nu+\frac{1}{2}}{x}\right)I_{\nu+1}(x)\right]. \tag{10}$$

Hence, the inequality $H_{\nu,k}(x) < I_\nu(x)$ is stronger than (3) for $k \geqq 2$ and $\nu > -\frac{1}{2}$.

The inequalities discussed here are all in the form of lower bounds of $I_\nu(x)$. An upper bound of $I_\nu(x)$ is derived as follows. Replace ν by $\nu + 1$ in (10), eliminate $I_{\nu+2}(x)$ from the bracket in the right-hand side of (10) by using (9), and make use of the positivity of the bracket. It follows that

$$I_\nu(x) < \frac{1 + 2(\nu+1)/x + 2(\nu+1)(\nu+\frac{3}{2})/x^2}{1 + (\nu+\frac{3}{2})/x} I_{\nu+1}(x) \tag{11}$$

for $x > 0$ and $\nu > -\frac{3}{2}$. Sharp versions of (11) are derived by making use of the inequalities $I_\nu(x) > H_{\nu,k}(x)$, $k \geqq 2$ or $I_\nu(x) > G_{\nu,k}(x)$, $k \geqq 3$. The general form for these upper bounds has not been found.

Luke [5] and Prohorov [6] have given inequalities for modified Bessel functions. These inequalities are weaker than those discussed here but have the virtue that the bounds for $I_\nu(x)$ are easily evaluated numerically.

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